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Rotation number for the one-dimensional Schrödinger operator with periodic singular potentials

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1. Introduction and main result

In this article, we survey the results in [13, 14, 15]. In those papers, we study the one-dimensional Schrödinger operatosr with singular potentials. In order to explain the motivation of our study, we describe its background. Such operators plays an important role in solid state physics (see [10]) and have been studied in numerous work [1, 2, 5, 6, 8, 11, 16, 17]. In 1931, Kronig and Penney introduced the Hamiltonians which is formally expressed as

$$L_1 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta(x - 2\pi l) \quad \text{in} \quad L^2(\mathbb{R}),$$

where $\delta(x)$ is the Dirac delta function at the origin and $\beta \in \mathbb{R} \setminus \{0\}$. The precise definition of L_1 is given through the boundary conditions on the lattice $2\pi\mathbb{Z}$ as follows.

$$(L_1 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

$$\mathrm{Dom}(L_1) = \left\{ y \in H^2(\mathbb{R} \setminus 2\pi\mathbb{Z}) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in 2\pi\mathbb{Z} \end{array} \right\},$$

where $H^2(D)$ denotes the Sobolev space of order 2 on an open set $D \subset \mathbb{R}$. This operator is the Hamiltonian for an electron in a one-dimensional crystal and is called Kronig-Penney Hamiltonian. The Dirac delta function is the most typical point interaction. The δ-interaction was widely generalized. In [5, 6], Gesztesy, Holden, and Kirsch inspired a new class of point interactions. They syudied the operator in $L^2(\mathbb{R})$ of the form

$$(L_2 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

$$\mathrm{Dom}(L_2) = \left\{ y \in H^2(\mathbb{R} \setminus 2\pi\mathbb{Z}) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in 2\pi\mathbb{Z} \end{array} \right\}.$$

This operator has the formal expression

$$L_2 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta'(x - 2\pi l) \quad \text{in} \quad L^2(\mathbb{R}).$$

In [16] , Šeba found that the domain of any self-adjoint extension of $(-d^2/dx^2)|_{C_0^\infty(\mathbb{R}\setminus\{0\})}$ in $L^2(\mathbb{R})$ of coupled type is expressed as

$$\left\{ y \in H^2(\mathbb{R} \setminus \{0\}) \;\middle|\; \begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = cA \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix} \right\}$$

with $A \in SL(2,\mathbb{R})$, $c \in \mathbb{C}$, and $|c| = 1$, where $SL(2,\mathbb{R})$ denotes the special linear group (see also [2] and [1, Section K.1.4]). In [8], Hughes gave the Floquet-Bloch decomposition of the Schrödinger operator in $L^2(\mathbb{R})$ with generalized point interaction on a lattice $2\pi\mathbb{Z}$ defined as

$$(L_3 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

$$\mathrm{Dom}(L_3) = \left\{ y \in H^2(\mathbb{R} \setminus 2\pi\mathbb{Z}) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = cA \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in 2\pi\mathbb{Z} \end{array} \right\}.$$

These backgrounds motivate us to study the spectra of the one-dimensional Schrödinger operators with periodic generalized point interactions.

To define the operators, we introduce notations. We fix $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Let $0 = \kappa_0 < \kappa_1 < \cdots < \kappa_n = 2\pi$ be a partition of the interval $(0, 2\pi)$. We put $\Gamma_j = \{\kappa_j\} + 2\pi\mathbb{Z}$ for $j = 1, 2, \dots, n$, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$. For $\{\theta_j\}_{j=1}^n \subset \mathbb{R}$ and $\{A_j\}_{j=1}^n \subset SL_2(\mathbb{R})$, we define the one-dimensional Schrödinger operator $H = H(\theta_1, \theta_2, \dots, \theta_n, A_1, A_2, \dots, A_n)$ in $L^2(\mathbb{R})$ as follows.

$$(Hy)(x) = -y''(x), \quad x \in \mathbb{R} \setminus \Gamma, \tag{1.1}$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\mathbb{R} \setminus \Gamma) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j, \quad j = 1, 2, \dots, n \end{array} \right\}. \tag{1.2}$$

This operator H is self-adjoint (see [13, Proposition 2.1]). Since the spectrum of H is independent of $\{\theta_j\}_{j=1}^n \subset \mathbb{R}$ (see [14, Proposition 1.1(e)]), we may put

$$\theta_1 = \theta_2 = \cdots = \theta_n = 0,$$

which does not cause any loss of generality. Since H has 2π-periodic point interactions, the spectrum of H has the band structure. According to the Floquet-Bloch theory, we label each band of the spectrum of H. For $j \in \mathbb{N}$, we designate the jth band of $\sigma(H)$ as

$$B_j = [\lambda_{2j-2}, \lambda_{2j-1}]. \tag{1.3}$$

The sequence $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{R}$ satisfies the inequalities

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \cdots \le \lambda_{2j-2} < \lambda_{2j-1} \le \lambda_{2j} < \cdots \to \infty.$$

So, the consequtive bands B_j and B_{j+1} are separated by an open interval

$$G_j := (\lambda_{2j-1}, \lambda_{2j}),$$

which is called the jth gap of $\sigma(H)$.

In [13, 14, 15], we mainly dealt with two problems. One of the problems is to give a characterization of the band edges of $\sigma(H)$ by the *rotation number*. The other is to determine the indices of the absent spectral gaps in a class of H.

We quote the main theorem in [14]. For this purpose, we introduce the rotation number. First, we consider the Schrödinger equation

$$-y''(x,\lambda) = \lambda y(x,\lambda), \quad x \in \mathbb{R} \setminus \Gamma, \tag{1.4}$$

$$\begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix}, \quad x \in \Gamma_j, \quad j = 1, 2, \ldots, n, \tag{1.5}$$

where λ is a real parameter. We define the Prüfer transform of a nontrivial solution $y(x,\lambda)$ to (1.4) and (1.5) as follows. Let (r,ω) be the polar coordinates of (y, y'):

$$y = r\sin\omega, \quad y' = r\cos\omega.$$

Then we call the function $\omega = \omega(x,\lambda)$ the Prüfer transform of $y(x,\lambda)$. For each $j = 1, 2, \cdots, n$, we write

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}. \tag{1.6}$$

Then, $\omega(x,\lambda)$ satisfies the equation

$$\omega'(x,\lambda) = \cos^2\omega(x,\lambda) + \lambda\sin^2\omega(x,\lambda), \quad x \in \mathbb{R} \setminus \Gamma \tag{1.7}$$

as well as the boundary conditions

$$\begin{aligned} &\sin\omega(x+0,\lambda)(c_j\sin\omega(x-0,\lambda) + d_j\cos\omega(x-0,\lambda)) \\ &\quad = \cos\omega(x+0,\lambda)(a_j\sin\omega(x-0,\lambda) + b_j\cos\omega(x-0,\lambda)), \end{aligned} \tag{1.8}$$

$$\operatorname{sgn}(\sin\omega(x+0,\lambda)) = \operatorname{sgn}(a_j\sin\omega(x-0,\lambda) + b_j\cos\omega(x-0,\lambda)), \tag{1.9}$$

$$\operatorname{sgn}(\cos\omega(x+0,\lambda)) = \operatorname{sgn}(c_j \sin\omega(x-0,\lambda) + d_j \cos\omega(x-0,\lambda)) \tag{1.10}$$

for $x \in \Gamma_j$ and $j = 1, 2, \ldots, n$, where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

To determine the principal value of $\omega(x+0,\lambda)$ by the boundary conditions (1.8), (1.9), and (1.10), we must select a branch of $\omega(x+0,\lambda)$ for $x \in \Gamma$. We choose the branch of $\omega(x+0,\lambda)$ as

$$\omega(x+0,\lambda) - \omega(x-0,\lambda) \in [-\pi, \pi) \quad \text{for} \quad x \in \Gamma. \tag{1.11}$$

Thanks to this selection, $\omega(x+0,\lambda)$ is uniquely determined. We pick $\omega_0 \in \mathbb{R}$. Let $\omega = \omega(x, \lambda, \omega_0)$ be the solution of (1.7) – (1.10) subject to the initial condition

$$\omega(+0,\lambda) = \omega_0. \tag{1.12}$$

We define the rotation number of H as

$$\rho(\lambda) = \lim_{n\to\infty} \frac{\omega(2n\pi + 0, \lambda, \omega_0) - \omega_0}{2n\pi}. \tag{1.13}$$

We recall (1.3). In [14], we proved the following theorem which relates $\rho(\lambda)$ to the spectrum of H.

Theorem 1.1. *The following statements* (a), (b), *and* (c) *hold true.*
(a) *The limit on the right-hand side of* (1.13) *exists and is independent of the initial value* ω_0.
(b) *The function* $\rho(\lambda)$ *is non-decreasing on* $\mathbb{R}$.
(c) *We put*

$$l = \sharp\{j \in \{1, 2, \ldots, n\} \mid (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0)\}, \tag{1.14}$$

where $\sharp A$ *stands for the number of the elements of* A *for a finite set* A. *Then, for* $j \in \mathbb{N}$, *we have*

$$\lambda_{2j-2} = \max\left\{\lambda \in \mathbb{R} \;\middle|\; \rho(\lambda) = \frac{j-1}{2} - \frac{l}{2}\right\}, \tag{1.15}$$

$$\lambda_{2j-1} = \min\left\{\lambda \in \mathbb{R} \;\middle|\; \rho(\lambda) = \frac{j}{2} - \frac{l}{2}\right\}. \tag{1.16}$$

We note that (1.15) and (1.16) critically depend on the choice of the branch of $\omega(x + 0, \lambda)$ for $x \in \Gamma$ (see [14, Section 4]).

The rotation number has a close relationship to the *density of states.* In order to see that, we introduce the density of states for H. For $k \in \mathbb{N}$, we put $I_k = \Gamma^c \cap (0, 2\pi k)$. Let us introduce the generalized Kronig-Penney Hamiltonian in $L^2((0,2\pi k))$ with the Dirichlet boundary conditions

$$y(+0) = y(2\pi k - 0) = 0.$$

We define the operator $H_{2\pi k,D}$ as

$$(H_{2\pi k,D}y)(x) = -y''(x), \quad x \in I_k,$$

$$\mathrm{Dom}(H_{2\pi k,D}) = \left\{ y \in H^2(\mathbb{R}\setminus\Gamma) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j \cap (0,2\pi k), \quad j = 1,2,\ldots,n, \\ y(+0) = y(2\pi k - 0) = 0 \end{array} \right\}.$$

For $n \in \mathbb{N} \cup \{0\}$, let $\lambda_{k,n}$ be the $(n+1)$st eigenvalue of $H_{2\pi k,D}$. Put

$$\nu(k,\lambda) = \sharp\{n \in \mathbb{N}\cup\{0\} \mid \ \lambda_{k,n} \le \lambda\}.$$

Then we have the following theorem.

Theorem 1.2. *We have*

$$\lim_{k\to\infty} \frac{\nu(k,\lambda)}{2\pi k} = \frac{\rho(\lambda)}{\pi} + \frac{l}{2\pi}. \tag{1.17}$$

In the physics literatures, the left-hand side of (1.17) is refered to as the density of states. We will give the outline of the proof of Theorem 1.2 in Section 2; the complete proof is found in [14].. On the other hand, we did not describe Theorem 1.2 in [14]. So, we give the complete proof of it in Section 2.

Our study [14] is also motivated by the works [9, 12], which we recall below. Johnson and Moser found that the rotation number for the one-dimensional Schrödinger operators with almost periodic potentials has a close relation to its spectrum. They dealt with the Schrödinger operator $L = -d^2/dx^2 + q(x)$, where q is an almost periodic function with a frequency module $\mathcal{M}$. They proved that the rotation number $\alpha(\lambda)$ for L exists and defines a continuous function in $\{\lambda \in \mathbb{C} \mid \ \mathrm{Im}\lambda \le 0\}$. Furthermore, $\alpha(\lambda)$ is constant in an open interval I in a spectral gap and $2\alpha(\lambda) \in \mathcal{M}$ for $\lambda \in I$. In the special case where q is periodic of period 2π, they found that the jth band $\tilde{B}_j$ of $\sigma(L)$ is expressed as

$$\tilde{B}_j = \overline{\left\{\lambda \;\middle|\; \frac{j-1}{2} < \alpha(\lambda) < \frac{j}{2}\right\}} \tag{1.18}$$

for $j \in \mathbb{N}$. This means that

$$\tilde{\lambda}_{2j-2} = \max\left\{\lambda \in \mathbb{R} \;\middle|\; \alpha(\lambda) = \frac{j-1}{2}\right\},$$

$$\tilde{\lambda}_{2j-1} = \min\left\{\lambda \in \mathbb{R} \;\middle|\; \alpha(\lambda) = \frac{j}{2}\right\},$$

where $\tilde{B}_j = [\tilde{\lambda}_{2j-2}, \tilde{\lambda}_{2j-1}]$. Let $N(x,\lambda)$ be the number of the zeroes in $[0,x]$ of a nontrivial solution to $(L\varphi)(x) = \lambda\varphi(x)$. Then they described that

$$\lim_{x\to\infty} \frac{N(x,\lambda)}{x} = \lim_{x\to\infty} \frac{\nu(x,\lambda)}{x} = \frac{\alpha(\lambda)}{\pi},$$

where $\nu = \nu(x,\lambda)$ is the number of eigenvalues of $(Ly)(x,\lambda) = \lambda y(x,\lambda)$ in $[0,x]$ with the boundary conditions $y(0) = y(x) = 0$. In contrast to these results, our theorems involve the number of the interactions in the fundamental region.

Next, we introduce the results in [13, 15]. The aim of those papers is to determine the indices of the absent spectral gaps of $H(\theta_1, \theta_2, A_1, A_2)$. In [13], we dealt with the case where

$$A_1, A_2 \in SO(2) \setminus \{E, -E\}, \tag{1.19}$$

E being the 2×2 unit matrix. We put

$$A_j = \begin{pmatrix} \cos\gamma_j & -\sin\gamma_j \\ \sin\gamma_j & \cos\gamma_j \end{pmatrix} \quad \text{and} \quad \gamma_j \in (0,\pi)\cup(\pi,2\pi)$$

for $j = 1,2$. We define

$$\Lambda = \{m \in \mathbb{N} | \quad G_m = \emptyset\}.$$

In [13], we have the following theorem.

Theorem 1.3. *Adopt the assumption* (1.19). *Let* $\kappa_1 \neq \pi$.
(a) *Suppose that* $\gamma_1 - \gamma_2 \not\equiv 0$ *and* $\gamma_1 + \gamma_2 \not\equiv 0 \pmod{\pi}$. *Then we have*

$$\Lambda = \emptyset.$$

(b) *Suppose that* $\gamma_1 + \gamma_2 \equiv 0 \pmod{\pi}$. *Then we have*

$$\Lambda = \begin{cases} \{3\} & \text{if } \frac{\kappa_1}{\pi} \notin \mathbb{Q}, \\ \{3\}\cup\{pk+1 | \quad k \in \mathbb{N}\} & \text{if } \frac{\kappa_1}{2\pi} = \frac{q}{p}, \quad (p,q)\in\mathbb{N}^2, \quad \text{and} \quad \gcd(p,q) = 1. \end{cases}$$

(c)*Assume that* $\gamma_1 - \gamma_2 \equiv 0$ *and* $\gamma_1 + \gamma_2 \not\equiv 0 \pmod{\pi}$. *We put* $\eta_j = \pi^2 j^2/4(\pi-\kappa_1)^2$ *for* $j \in \mathbb{N}$. *Then it holds that*

$$\bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \left\{ \eta_j \,\middle|\, -2\left(\sqrt{\eta_j} + \frac{1}{\sqrt{\eta_j}}\right)^{-1} \cot\kappa_1\sqrt{\eta_j} = \tan\gamma_1 \text{ and } j \in \mathbb{N} \right\}.$$

In [15], we dealt with the case where

$$A_1 A_2 = \pm E \quad \text{and} \quad A_1, A_2 \in SL(2,\mathbb{R}) \setminus \{E, -E\}. \tag{1.20}$$

For convenience we rewrite the elements of A_1 as

$$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have the following theorem [15, Theorem 1.2].

Theorem 1.4. *Adopt the assumption* (1.20). *Let* $\kappa_1 \neq \pi$.
(a) *Assume that* $\kappa_1/\pi \notin \mathbb{Q}$. *Then we have*

$$\Lambda = \begin{cases} \{k+1\} & \text{if} \quad d = a, \quad b \neq 0, \quad -c/b = k^2/4 \quad \text{for some } k \in \mathbb{N}, \\ \emptyset & \text{otherwise.} \end{cases}$$

(b) *Suppose that* $\kappa_1/2\pi = q/p$, $(p,q) \in \mathbb{N}^2$, *and* $\gcd(p,q) = 1$. *Then we have*

$$\Lambda = \begin{cases} \{pj \mid \ j \in \mathbb{N}\} & \text{if} \quad b = 0, \\ \{1 + pj \mid \ j \in \mathbb{N}\} \cup \{1+k\} & \text{if} \quad d = a, \quad b \neq 0, \quad -c/b = k^2/4, \\ & \qquad \text{for some } k \in \mathbb{N}, \quad k \not\equiv 0 \pmod{p}, \\ \{1 + pj \mid \ j \in \mathbb{N}\} & \text{otherwise.} \end{cases}$$

Using Theorem 1.1, we can newly get a theorem. We discuss the spectral gaps of the Schrödinger operator formally expressed as

$$L_4 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} \left(\beta_1 \delta(x - \kappa_1 - 2\pi l) + \beta_2 \delta'(x - 2\pi l) \right),$$

where $\kappa_1 \in (0, 2\pi)$ and $\beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}$ are parameters. In our notations this operator is expressed as $L_4 = H(0,0,M_1,M_2)$, where

$$M_1 = \begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix}.$$

We have the following theorem for the operator L_4.

Theorem 1.5. *We suppose that* $\kappa_1 \neq \pi$ *and*

$$(\beta_1, \beta_2) \notin \left\{ \left(\frac{n\pi}{|\pi - \kappa_1|} \tan \frac{\kappa_1 n \pi}{2|\pi - \kappa_1|}, -\frac{4|\pi - \kappa_1|}{n\pi} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|} \right) \middle| \ n \in \mathbb{N} \right\}.$$

Then we have the following statements (i) *and* (ii).
(i) *If either* $\kappa_1 \notin \{\pi/2, 3\pi/2\}$ *or* $\beta_1 \neq \beta_2$ *holds, then*

$$\Lambda = \emptyset.$$

(ii) *If* $\kappa_1 \in \{\pi/2, 3\pi/2\}$ *and* $\beta_1 = \beta_2$, *then*

$$\Lambda = \begin{cases} \{2\} & \text{if} \quad \beta_1 > 0, \\ \{3\} & \text{if} \quad \beta_1 < 0. \end{cases}$$

The study of L_4 is motivated by the work [17]. In [17], Yoshitomi investigated the spectral gaps of the operators

$$P_0 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} \left(\beta_1 \delta(x - \kappa - 2\pi l) + \beta_2 \delta(x - 2\pi l) \right) \quad \text{in} \quad L^2(\mathbb{R}),$$

and

$$P_1 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} \left(\beta_1 \delta'(x - \kappa - 2\pi l) + \beta_2 \delta'(x - 2\pi l) \right) \quad \text{in} \quad L^2(\mathbb{R}),$$

where $\kappa \in (0, 2\pi)$. For $j \in \mathbb{N}$ and $k \in \{0,1\}$, he described that $\sigma(P_k)$ has an absent gap if and only if $\beta_1 + \beta_2 = 0$ and $\kappa/\pi \in \mathbb{Q}$ hold. Furthermore, his theorems say that if $\beta_1 + \beta_2 = 0$ and $\kappa/2\pi = m/n$, $(n,m) \in \mathbb{N}^2$, and $\gcd(m,n) = 1$, then the jth gap of $\sigma(P_k)$ is absent if and only if $j - k \in n\mathbb{N}$. We prove Theorem 1.5 in Section 3.

2. Proof of Theorem 1.2 and 1.3

In this section, we describe the proof of Theorem 1.2 and 1.3. We recall (1.6). Let

$$q_j = \sharp\{k \in \{1,2,\dots,j\} | \quad (b_k < 0) \quad \text{or} \quad (b_k = 0, \quad d_k < 0)\},$$
$$q_0 = 0,$$

and

$$\eta_j = \begin{cases} \mathrm{Arctan}(b_j/d_j) - q_{j-1}\pi & \text{if} \quad b_j > 0, \quad d_j > 0, \\ \mathrm{Arctan}(b_j/d_j) + \pi - q_{j-1}\pi, & \text{if} \quad b_j > 0, \quad d_j < 0, \\ \pi/2 - q_{j-1}\pi & \text{if} \quad b_j > 0, \quad d_j = 0, \\ \mathrm{Arctan}(b_j/d_j) - \pi - q_{j-1}\pi, & \text{if} \quad b_j < 0, \quad d_j < 0, \\ \mathrm{Arctan}(b_j/d_j) - q_{j-1}\pi & \text{if} \quad b_j < 0, \quad d_j > 0, \\ -\pi/2 - q_{j-1}\pi & \text{if} \quad b_j < 0, \quad d_j = 0, \\ -q_{j-1}\pi & \text{if} \quad b_j = 0, \quad d_j > 0, \\ -\pi - q_{j-1}\pi & \text{if} \quad b_j = 0, \quad d_j < 0 \end{cases}$$

for $j = 1, 2, \dots, n$, where $\mathrm{Arctan}(x) \in (-\pi/2, \pi/2)$ for $x \in \mathbb{R}$. Since

$$q_j = \begin{cases} q_{j-1} + 1 & \text{if} \quad (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0), \\ q_{j-1} & \text{otherwise}, \end{cases}$$

we have

$$\eta_j \in [-q_j\pi, -q_j\pi + \pi) . \tag{2.1}$$

We pick a $\gamma \in (0, \pi)$ such that

$$\eta_j < -q_j\pi + \gamma \quad \text{for} \quad j = 1, 2, , \dots, n.$$

Then we have the following lemma.

Lemma 2.1. *There exists $\lambda_0 \in \mathbb{R}$ such that*

$$-\pi(q_j + pq_n) \leq \omega(\kappa_j + 2\pi p + 0, \lambda, \omega_0) \leq -\pi(q_j + pq_n) + \gamma$$

for any $p \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \dots, n$, $\lambda \leq \lambda_0$, and $\omega_0 \in [0, \gamma]$.

To prove this lemma, we recall a fundamental fact on the Prüfer transform from [3, Chapter 8, Theorem 2.1]. Let $c < d$. For $\beta \in [0, \pi)$, let $\theta = \theta(x, \lambda, c, \beta)$ be the solution to the initial value problem

$$\frac{d}{dx}\theta = \cos^2\theta + \lambda \sin^2\theta \quad \text{on} \quad \mathbb{R}, \tag{2.2}$$

$$\theta|_{x=c} = \beta. \tag{2.3}$$

Then, it holds that

$$\lim_{\lambda \to -\infty} \theta(d, \lambda, c, \beta) = 0. \tag{2.4}$$

Moreover, the function $\theta(d, \cdot, c, \beta)$ is strictly monotone increasing on $\mathbb{R}$.

We describe the outline of the proof of Lemma 2.1.

Outline of the proof of Lemma 2.1. We fix $\omega_0 \in [0,\gamma]$. First, we shall show the following statements by induction on $j = 1, 2, \dots, n$.

The limit $\beta_j := \lim_{\lambda\to-\infty} \omega(\kappa_j - 0, \lambda, \omega_0) \in \mathbb{R}$ exists, and we have $\beta_j = -q_{j-1}\pi$. (2.5)

The function $\omega(\kappa_j - 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$. (2.6)

It follows by (2.4) that (2.5) and (2.6) are valid for $j = 1$. We pick $m \in \{1, 2, \dots, n\}$, arbitrarily. Suppose that (2.5) and (2.6) hold for $j = m$. Then we can show that the limit

$$\alpha_m := \lim_{\lambda\to-\infty} \omega(\kappa_m + 0, \lambda, \omega_0)$$

exists and

$$\alpha_m = \eta_m. \tag{2.7}$$

By (1.8), we have

$$\tan\omega(\kappa_m + 0, \lambda, \omega_0) = \frac{a_m \tan\omega(\kappa_m - 0, \lambda, \omega_0) + b_m}{c_m \tan\omega(\kappa_m - 0, \lambda, \omega_0) + d_m}. \tag{2.8}$$

Combining the monotonicity of $\omega(\kappa_m - 0, \cdot, \omega_0)$ and $a_m d_m - b_m c_m = 1$ with (2.8), we find that $\omega(\kappa_m + 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$.

Since $\omega(\kappa_{m+1} - 0, \lambda, \omega_0) = \theta(\kappa_{m+1}, \lambda, \kappa_m, \omega(\kappa_m + 0, \lambda, \omega_0))$, (2.6) is valid for $j = m+1$. Using the monotonicity of $\omega(\kappa_m, \cdot, \omega_0)$, we infer that there exists $\lambda_m \in \mathbb{R}$ such that

$$-q_m\pi \le \omega(\kappa_m + 0, \lambda, \omega_0) \le -q_m\pi + \gamma \tag{2.9}$$

for $\lambda \le \lambda_m$. By the comparison theorem [3, Chapter 8] and (2.9), we have

$$\theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi) \le \omega(\kappa_{m+1} - 0, \lambda, \omega_0) < \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi + \gamma)$$

for $\lambda \le \lambda_m$. Since the equation (2.2) is π-periodic, we derive

$$\lim_{\lambda\to-\infty} \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi) = \lim_{\lambda\to-\infty} \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi + \gamma) = -q_m\pi,$$

so that

$$\beta_{m+1} = -q_m\pi.$$

So, we have proved (2.5) and (2.6) for $j = m+1$. Therefore, (2.5) and (2.6) are valid for $j = 1, 2, \dots, n$.

Put $\lambda_0 = \min_{1\le j\le n} \lambda_j$. We have

$$-\pi q_j \le \omega(\kappa_j + 0, \lambda, \omega_0) < -\pi q_j + \gamma \tag{2.10}$$

for $j = 1, 2, \dots, n$, and $\lambda \le \lambda_0$.

Using the comparison theorem and $\omega_0 \in [0,\gamma]$, we notice that

$$\omega(\kappa_j + 0, \lambda, 0) \le \omega(\kappa_j + 0, \lambda, \omega_0) \le \omega(\kappa_j + 0, \lambda, \gamma).$$

Therefore the estimate (2.10) is uniform with respect to $\omega_0 \in [0,\gamma]$.

Since the equations (1.6) – (1.9) is 2π-periodic with respect to x, we have the desired assertion from (2.10). □

Proof of Theorem 1.1. By a similar way to the proof of [7, Theorem 2.1], it follows that (a) and (b) hold. So, we have only to show the statement (c). We recall (1.14). Then, we notice that $q_n = l$. By Lemma 2.1, we have

$$-\pi pl \leq \omega(2\pi p + 0, \lambda, \omega_0) \leq -\pi pl + \gamma$$

for $0 \leq \omega_0 \leq \gamma$, $\lambda \leq \lambda_0$, and $p \in \mathbb{N}$. This together with (1.13) implies that

$$\lim_{\lambda \to -\infty} \rho(\lambda) = -\frac{l}{2}. \tag{2.11}$$

Combining (2.11) with the discussion in the proof of [4, Proposition 2.1.], we get the assertion (c). □

Proof of Theorem 1.3. By (2.5) and (2.6), we have

$$\lim_{\lambda \to -\infty} \omega(\kappa_j - 0, \lambda, \omega_0) = -q_{j-1}\pi,$$

and the function $\omega(\kappa_j - 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$. Since the equation (1.7) – (1.10) is 2π-periodic with respect to x, we have

$$\lim_{\lambda \to -\infty} \omega(2\pi p - 0, \lambda, \omega_0) = -q_{n-1}\pi - \pi(p-1)l$$

and the function $\omega(2\pi p - 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$ for $p \in \mathbb{N}$. Because of the monotonicity of $\omega(2\pi p - 0, \cdot, \omega_0)$, there exists $\lambda_{p,m} \in \mathbb{R}$ satisfying

$$\omega(2\pi p - 0, \lambda_{p,m}, \omega_0) = -\pi\{q_{n-1} + (p-1)l\} + m\pi$$

for each $m \in \mathbb{N}$. In a similar way to [3, Chapter 8, Theorem 2.1], we see that $\lambda_{p,m}$ is the $(m+1)$st eigenvalue of $H_{2\pi p, D}$.

We fix $\lambda \in \mathbb{R}$, arbitrarily. Define

$$m_p^* = \sharp\{m \in \mathbb{N} | \quad \lambda_{p,m} \leq \lambda\} + 1.$$

Then we have

$$\lambda_{p,m_p^*} \leq \lambda < \lambda_{p,m_p^*+1}.$$

By the monotonicity of $\omega(2\pi p - 0, \cdot, \omega_0)$, we have

$$-\pi\{q_{n-1} + (p-1)l\} + m_p^*\pi < \omega(2\pi p + 0, \lambda, \omega_0) < -\pi\{q_{n-1} + (p-1)l\} + (m_p^* + 1)\pi.$$

This inequality reduces

$$m_p^* < \frac{\omega(2\pi p + 0, \lambda, \omega_0)}{\pi} + q_{n-1} + (p-1)l < m_p^* + 1.$$

So we derive

$$m_p^* = \left[\frac{\omega(2\pi p + 0, \lambda, \omega_0)}{\pi}\right] + q_{n-1} + (p-1)l.$$

By the definition of $\gamma(p,\lambda)$ and m_p^*, we have

$$\gamma(p,\lambda) = m_p^* = \left[\frac{\omega(2\pi p+0,\lambda,\omega_0)}{\pi}\right] + q_{n-1} + (p-1)l. \tag{2.12}$$

On the other hand, we notice that

$$\begin{aligned} & \frac{\omega(2\pi p+0,\lambda,\omega_0)/\pi}{2p\pi} + \frac{q_{n-1}+(p-1)l-1}{2p\pi} \\ \leq & \frac{[\omega(2\pi p+0,\lambda,\omega_0)/\pi]}{2p\pi} + \frac{q_{n-1}+(p-1)l}{2p\pi} \\ \leq & \frac{\omega(2\pi p+0,\lambda,\omega_0)/\pi}{2p\pi} + \frac{q_{n-1}+(p-1)l}{2p\pi} \end{aligned} \tag{2.13}$$

Using (2.12), (2.13), and (1.11), we get (1.17). □

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. In the first place, we define the monodromy matrix. For this purpose, we consider the equations

$$-y''(x,\lambda) = \lambda y(x,\lambda), \quad x \in \mathbb{R}\setminus\Gamma, \tag{3.1}$$

$$\begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix} \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix}, \quad x \in \Gamma_1, \tag{3.2}$$

$$\begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix}, \quad x \in \Gamma_2, \tag{3.3}$$

where λ is real parameter. These equations have two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ which are uniquely determined by the initial conditions

$$y_1(+0,\lambda) = 1, \quad y_1'(+0,\lambda) = 0,$$

and

$$y_2(+0,\lambda) = 0, \quad y_2'(+0,\lambda) = 1,$$

respectively. Then, the monodromy matrix of (3.1) - (3.3) is defined as

$$M(\lambda) = \begin{pmatrix} y_1(2\pi+0,\lambda) & y_2(2\pi+0,\lambda) \\ y_1'(2\pi+0,\lambda) & y_2'(2\pi+0,\lambda) \end{pmatrix} \tag{3.4}$$

As described in [17, Lemma 4] (see also [13, 15]), we have

$$\mathcal{B} := \bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \{\lambda \in \mathbb{R} \mid \ M(\lambda) = E \quad \text{or} \quad M(\lambda) = -E\}.$$

Put

$$\tau = 2\pi - \kappa_1.$$

By a direct calculation, we get

$$\begin{aligned}
y_1(2\pi+0,\lambda) &= (1+\beta_1\beta_2)\cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} + \left(\frac{\beta_1}{\sqrt{\lambda}} - \beta_2\sqrt{\lambda}\right)\sin\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} \\
&\quad -\beta_2\sqrt{\lambda}\cos\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda} - \sin\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda}, \qquad (3.5)\\
y_1'(2\pi+0,\lambda) &= \beta_1\cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} - \sqrt{\lambda}\sin\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} - \sqrt{\lambda}\cos\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda}, \qquad (3.6)\\
y_2(2\pi+0,\lambda) &= \beta_2\cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\sin\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} \\
&\quad +\frac{1+\beta_1\beta_2}{\sqrt{\lambda}}\cos\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda} + \left(\frac{\beta_1}{\lambda} - \beta_2\right)\sin\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda}, \qquad (3.7)\\
y_2'(2\pi+0,\lambda) &= \cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} + \frac{\beta_1}{\sqrt{\lambda}}\cos\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda} - \sin\tau\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda}. \qquad (3.8)
\end{aligned}$$

In order to establish Theorem 1.5, we show the following theorem.

Theorem 3.1. *We suppose that $\kappa_1 \neq \pi$ and*

$$(\beta_1,\beta_2) \notin \left\{\left(\frac{n\pi}{|\pi-\kappa_1|}\tan\frac{\kappa_1 n\pi}{2|\pi-\kappa_1|}, -\frac{4|\pi-\kappa_1|}{n\pi}\tan\frac{\kappa_1 n\pi}{2|\pi-\kappa_1|}\right) \middle|\ n\in\mathbb{N}\right\}. \qquad (3.9)$$

Then we have the following statements (i) *and* (ii).
(i) *If either $\kappa_1 \notin \{\pi/2, 3\pi/2\}$ or $\beta_1 \neq \beta_2$ holds, then we have*

$$\mathcal{B} = \emptyset.$$

(ii) *If $\kappa_1 \in \{\pi/2, 3\pi/2\}$ and $\beta_1 = \beta_2$, then we have*

$$\mathcal{B} = \{1\}.$$

We prove this theorem by using the following lemma.

Lemma 3.2. *Assume that $\kappa_1 \neq \pi$ and $M(\lambda) = \pm E$. Then we have the following statements.*
(i) *If $\lambda \neq -\beta_1/\beta_2$, then $\lambda = \beta_2/\beta_1$ and $\cos\kappa_1\sqrt{\lambda} = \cos\tau\sqrt{\lambda} = 0$.*
(ii) *If $\lambda = -\beta_1/\beta_2$, then there exists $n \in \mathbb{N}$ such that*

$$\beta_1 = \frac{n\pi}{|\pi-\kappa_1|}\tan\frac{\kappa_1 n\pi}{2|\pi-\kappa_1|},$$

and

$$\beta_2 = -\frac{4|\pi-\kappa_1|}{n\pi}\tan\frac{\kappa_1 n\pi}{2|\pi-\kappa_1|}.$$

Proof. We suppose that $M(\lambda) = \pm E$. We first show that $\lambda \neq 0$. We have

$$M(0) = \begin{pmatrix} 1+\beta_1\beta_2+\beta_1\tau & \beta_2+\tau+(1+\beta_1\beta_2)\kappa_1+\beta_1\kappa_1\tau \\ 0 & 1+\beta_1\kappa_1 \end{pmatrix}.$$

This means $M(0) \neq \pm E$ because of $1+\beta_1\kappa_1 \neq 1$. This is why $\lambda \neq 0$.

Since $M(\lambda) = \pm E$, we have

$$y_1(2\pi+0,\lambda) - y_2'(2\pi+0,\lambda) = y_1'(2\pi+0,\lambda) = y_2(2\pi+0,\lambda) = 0.$$

By $\{y_1'(2\pi+0,\lambda)/\lambda + y_2(2\pi+0,\lambda)\}\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} = 0$, it turns out that

$$\left(\frac{\beta_1}{\sqrt{\lambda}}+\beta_2\sqrt{\lambda}\right)\cos\tau\sqrt{\lambda}\cos^2\kappa_1\sqrt{\lambda}+\beta_1\beta_2\cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda} \\ +\left(\frac{\beta_1}{\sqrt{\lambda}}-\beta_2\sqrt{\lambda}\right)\sin\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda} = 0. \quad (3.10)$$

On the other hand, it follows by $(y_1(2\pi+0,\lambda) - y_2'(2\pi+0,\lambda))\sin\kappa_1\sqrt{\lambda} = 0$ that

$$\beta_1\beta_2\cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda}+\left(\frac{\beta_1}{\sqrt{\lambda}}-\beta_2\sqrt{\lambda}\right)\sin\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda}\sin\kappa_1\sqrt{\lambda} \\ -\left(\beta_2\sqrt{\lambda}+\frac{\beta_1}{\sqrt{\lambda}}\right)\cos\tau\sqrt{\lambda}\sin^2\kappa_1\sqrt{\lambda} = 0. \quad (3.11)$$

Substituting (3.11) from (3.10), we have

$$\left(\frac{\beta_1}{\sqrt{\lambda}}+\beta_2\sqrt{\lambda}\right)\cos\tau\sqrt{\lambda} = 0,$$

namely

$$\frac{\beta_1}{\sqrt{\lambda}}+\beta_2\sqrt{\lambda} = 0 \quad \text{or} \quad \cos\tau\sqrt{\lambda} = 0. \quad (3.12)$$

We show the statement (i). We suppose that $\lambda \neq -\beta_1/\beta_2$. Then it follows by (3.12) that $\cos\tau\sqrt{\lambda} = 0$. This combined with $\lambda \neq 0$ and $y_1'(2\pi+0,\lambda) = 0$ means $\cos\kappa_1\sqrt{\lambda} = 0$. Substituting $\cos\kappa_1\sqrt{\lambda} = \cos\tau\sqrt{\lambda} = 0$ for $y_2(2\pi+0,\lambda) = 0$, we have $\lambda = \beta_2/\beta_1$. Therefore we get (i).

Next, we show the statement (ii). We suppose that $\lambda = -\beta_1/\beta_2$. Then we have $\beta_1/\sqrt{\lambda}+\beta_2\sqrt{\lambda} = 0$. Substituting $\beta_1/\sqrt{\lambda} = -\beta_2\sqrt{\lambda}$ for $(y_1(2\pi+0,\lambda) - y_2(2\pi+0,\lambda))/\beta_2 = 0$, we have

$$\sin\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} - \frac{\beta_1}{2\sqrt{\lambda}}\cos\tau\sqrt{\lambda}\cos\kappa_1\sqrt{\lambda} = 0. \quad (3.13)$$

We prove $\cos\kappa_1\sqrt{\lambda} \neq 0$ by contradiction. Seeking a contradiction, we assume $\cos\kappa_1\sqrt{\lambda} = 0$. Then it follows by $y_1'(2\pi+0,\lambda) = 0$ and $\lambda \neq 0$ that $\cos\tau\sqrt{\lambda} = 0$. Substituting $\cos\kappa_1\sqrt{\lambda} = \cos\tau\sqrt{\lambda} = 0$ for $y_2(2\pi+0,\lambda) = 0$, we have $\lambda = \beta_1/\beta_2$. This contradicts $\lambda = -\beta_1/\beta_2$. Therefore we have $\cos\kappa_1\sqrt{\lambda} \neq 0$.

By (3.13) and $\cos\kappa_1\sqrt{\lambda} \neq 0$, it follows that

$$\sin\tau\sqrt{\lambda} = \frac{\beta_1}{2\sqrt{\lambda}}\cos\tau\sqrt{\lambda}. \tag{3.14}$$

Inserting $\beta_1/\lambda = -\beta_2$ and (3.14) into (3.6), we have

$$\sin\kappa_1\sqrt{\lambda} = \frac{\beta_1}{2\sqrt{\lambda}}\cos\kappa_1\sqrt{\lambda}. \tag{3.15}$$

By (3.14) and (3.15), it turns out that $\sin(\tau-\kappa_1)\sqrt{\lambda} = 0$. This implies that $\beta_1/\beta_2 < 0$ because of $\lambda = -\beta_1/\beta_2$ and $\tau - \kappa_1 \neq 0$. Substituting $\lambda = -\beta_1/\beta_2$ and $\tau = 2\pi - \kappa_1$ for $\sin(\tau-\kappa_1)\sqrt{\lambda} = 0$, we obtain

$$\sin 2(\pi-\kappa_1)\sqrt{-\frac{\beta_1}{\beta_2}} = 0.$$

Namely, there exists $n \in \mathbb{N}$ such that

$$-\frac{\beta_1}{\beta_2} = \frac{n^2}{4(\pi-\kappa_1)^2}. \tag{3.16}$$

On the other hand, Equation (3.15) means

$$\beta_1 = \frac{n\pi}{|\pi-\kappa_1|}\tan\frac{\kappa_1 n\pi}{2|\pi-\kappa_1|}.$$

This combined with (3.16) implies

$$\beta_2 = -\frac{4|\pi-\kappa_1|}{n\pi}\tan\frac{\kappa_1 n\pi}{2|\pi-\kappa_1|}.$$

□

Next, we show Theorem 3.1.

Proof of Theorem 3.1. We suppose $\kappa_1 \neq \pi$ and (3.9). We define

$$S = \begin{cases} \{\beta_2/\beta_1\} & \text{if} \quad \cos\kappa_1\sqrt{\beta_2/\beta_1} = \cos\tau\sqrt{\beta_2/\beta_1} = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, Lemma 3.2 says

$$S \subset \mathcal{B}.$$

Since $S \supset \mathcal{B}$, we have $\mathcal{B} = \emptyset$ if $S = \emptyset$. Next we consider the case where $S \neq \emptyset$. We have $S = \{\xi\}$, where $\xi = \beta_2/\beta_1$. Since

$$M(\xi) = -\sin\kappa_1\sqrt{\xi}\sin\tau\sqrt{\xi}\begin{pmatrix} 1 & \beta_2 - \frac{\beta_1}{\lambda} \\ 0 & 1 \end{pmatrix},$$

and

$$\beta_2 - \frac{\beta_1}{\lambda} = \beta_2 - \frac{\beta_1}{\frac{\beta_2}{\beta_1}} = \frac{(\beta_2 - \beta_1)(\beta_2 + \beta_1)}{\beta_2},$$

$M(\xi) = \pm E$ is equivalent to

$$\beta_2 - \beta_1 = 0 \quad \text{or} \quad \beta_2 + \beta_1 = 0, \tag{3.17}$$

whence $\xi \in \mathcal{B}$ if and only if (3.17) holds. This together with $\{\xi\} = S \supset \mathcal{B}$ implies that

$$\mathcal{B} = \begin{cases} \{\xi\} & \text{if} \quad \beta_2 - \beta_1 = 0 \quad \text{or} \quad \beta_2 + \beta_1 = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $\beta_1 + \beta_2 = 0$, then we have $S = \emptyset$, so that $\mathcal{B} = \emptyset$. If $\beta_2 - \beta_1 = 0$, then we obtain

$$\mathcal{B} = S = \begin{cases} \{1\} & \text{if} \quad \kappa_1 = \frac{\pi}{2}, \frac{3}{2}\pi, \\ \emptyset & \text{otherwise.} \end{cases}$$

□

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. Theorem 3.1 (i) directly follows Theorem 1.5 (i). So, our last work is to prove (ii). We suppose $\kappa_1 - \pi/2$ and $\beta_1 = \beta_2$. Then, Theorem 3.1 (ii) reads $\mathcal{B} = \{1\}$.

We calculate the rotation number $\rho(1)$. Substituting $\lambda = 1$ for (1.7), we have

$$\frac{d}{dx}\omega(x, \lambda) = 1, \quad x \in \mathbb{R} \setminus \Gamma. \tag{3.18}$$

Since the rotation number is independent of the initial value ω_0, we may put $\omega_0 = 0$. Equation (3.18) means $\omega(\kappa_1 - 0, 1, 0) = \pi/2$. It follows from (1.8)–(1.11) that

$$\omega(\kappa_1 + 0, 1, 0) = \begin{cases} \text{Arctan}\left(\frac{1}{\beta_1}\right) & \text{if} \quad \beta_1 > 0, \\ \pi + \text{Arctan}\left(\frac{1}{\beta_1}\right) & \text{if} \quad \beta_1 < 0. \end{cases}$$

Using Equation (3.18) again, we have

$$\omega(2\pi - 0, 1, 0) = \begin{cases} \text{Arctan}\left(\frac{1}{\beta_1}\right) + (2\pi - \kappa_1) & \text{if} \quad \beta_1 > 0, \\ \pi + \text{Arctan}\left(\frac{1}{\beta_1}\right) + (2\pi - \kappa_1) & \text{if} \quad \beta_1 < 0. \end{cases}$$

Using (1.8)–(1.11) in the case where $x = 2\pi - 0$, we have $\omega(2\pi + 0, 1, 0) = 2\pi$. Since the equation (1.7) is π-periodic in ω, we have $\omega(2\pi t + 0, 1, 0) = 2\pi t$ for $t \in \mathbb{N}$. Therefore we have $\rho(1) = 1$.

We recall (1.14). Since

$$l = \begin{cases} 1 & \text{if} \quad \beta_1 > 0, \\ 0 & \text{if} \quad \beta_1 < 0, \end{cases}$$

then we arrive at the goal owing to Theorem 1.1.

In a similar way, we obtain (ii) in the case where $\kappa_1 = 3\pi/2$ and $\beta_1 = \beta_2$. □

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